Abstract

Achieving cost targets is a major concern for business managers. In this paper, we consider two risk management criteria for a production-inventory system under a cost target: Probability of Loss and Expected Loss. We study two models with stochastic demand and production: The unit stockout cost model and the backlogging cost rate model. We analyze a limited-information setting that is an excellent approximation to a full-information setting. We discover that the optimal inventory decisions for minimizing probability of loss are identical for both models, and that the optimal inventory decisions for minimizing expected loss share a similar structure for both models. In addition, we investigate inventory decisions when minimizing the expected cost subject to a probability of loss constraint. Extension to a generally-distributed unit production time is also explored. We provide comparative statics and managerial insights of value to loss-aware managers.

Keywords: risk management, cost target, inventory, demand-capacity interaction
1. Introduction and Literature Review

Motivated by management by objectives (Drucker, 1954; Greenwood, 1981) and the cost targeting strategy (Shank and Fisher, 1999; Graves and Willems, 2005), firms usually set a target for their operating cost and seek to maintain their realized cost under such target, particularly for cost centers such as factories (Iranmanesh and Thomson, 2008; Arreola-Risa and Keys, 2013; Lim and Wang, 2017; Kerkhove and Vanhoucke, 2017). A similar emphasis on meeting cost targets can be found in regulator-determined operating cost targets for utility companies (Szabó et al., 2018). To measure the extent to which the realized cost exceeds the cost target, partial moment criteria have been used extensively in theory and practice (Tapiero, 2005). In this paper, we examine two partial moment criteria: Probability of loss and expected loss. The probability of loss measures the likelihood of the realized cost exceeding a pre-defined cost target, while the expected loss measures the expected magnitude of the realized cost exceeding the cost target. Our goal is to understand the role that inventory decisions play in achieving a pre-specified cost target, when the objective is to minimize the probability of loss, the expected loss, or the expected cost subject to a probability of loss constraint. This goal was inspired by our work in a real-world supply chain design project (Arreola-Risa and Keys, 2013).

1.1. Studies related to probability of loss

High costs are frequent targets for performance improvement. In business practice, a Japanese manufacturer breaks its operating cost down to various components and document the probability of cost problems for managerial accountability (Yoshikawa et al., 1995). Some researchers propose to maximize the probability of exceeding a profit aspiration level (Lau, 1980; Parlar and Weng, 2003; Shi et al., 2010; He and Khouja, 2011), which is equivalent to minimizing the probability of loss. Peter and Ryan (1976) consider probability of loss as an important criteria in consumers’ automobile brand selection. Using experiments, Unser (2000) finds empirical support for minimizing the probability of loss among various partial moment criteria. Eeckhoudt and Gollier (2005) investigate the optimal level of effort to reduce the probability of loss-occurrence for a prudent decision-maker. In newsvendor models, probability of loss is considered for a price-setting newsvendor (Jammernegg
and Kischka, 2013a) and a classical newsvendor problem (Jammernegg and Kischka, 2013b). Probability of loss is also considered important in supply chain disaster response (Knemeyer et al., 2009). In this paper, we consider probability of loss in combination with inventory and production, which differs from and complements the settings reviewed lines above.

1.2. Studies related to expected loss

Different from likelihood measures such as probability of loss, the expected loss beyond a threshold can serve as a useful performance-benchmarking tool as it measures the magnitude of exceeding the threshold, which is particularly suitable for loss-averse managers (Abeler et al., 2011; Herweg, 2013) and managers with a cost target in mind (Lim and Wang, 2017). Various supply chain management studies consider expected losses due to loss aversion (Vipin and Amit, 2017; Wang and Wang, 2018). In particular, Fleischhacker and Fok (2015) prove that reducing demand uncertainty can reduce expected loss in an inventory system. In this paper, we consider the minimization of expected loss for a single-product production-inventory system, which extends all the research just cited.

1.3. Studies with supply sources modeled as queuing systems

Kaplan (1970) is the first to characterize the optimal inventory policy for a multi-stage inventory system with exogenous lead times, whereas subsequent studies model the supply sources as a single-server queue (Karmarkar, 1987), serial queues (Lee and Zipkin, 1992), or queuing networks (Zipkin, 1986; Lee and Zipkin, 1995). Similar to this paper, Arreola-Risa (1996) analyzes an infinite-horizon, multi-item, supply chain model with a base-stock inventory policy in which the lead time depends on both product demand and manufacturing capacity. Caldentey and Wein (2003) design a coordinating contract for a supply chain modeled as an $M/M/1$ make-to-stock queue where inventory is owned by the retailer and the production system is owned by the supplier. More recently, Savaşaneril et al. (2010) consider a base-stock inventory system characterized by lead-time sensitive Poisson demand and exponentially distributed service times, whereas Benjaafar et al. (2010) consider the optimal control of a production-inventory system with a single product where items are produced one unit at a time. Arreola-Risa and Keblis (2013) study a production-inventory system and characterize the conditions for zero inventory. Otten et al. (2016) consider a two-echelon production-inventory system with a central supplier connected to production systems at several locations, each with a local inventory. Song et al. (2016) obtain optimal ordering policies for the discounted or long-run average cost of a single-product, two-source inventory system with
Poisson demand and backlogging. Hum et al. (2018) consider a single- and multi-server multi-stage serial supply chain and find a closed-form expression for the fulfillment time distribution. Elhafsi and Hamouda (2018) coordinate pricing, production scheduling and inventory allocation for a manufacturer selling a product through long-term and spot markets with exponentially-distributed production times. The just cited studies minimize expected cost or maximize expected profit. Despite also considering a production-inventory system with endogeneous lead times, our paper instead focuses on the probability and the magnitude of losses.

### 1.4. Contributions of this paper

First, this paper considers the objectives of minimizing probability of loss and expected loss in a production-inventory system with endogenous lead times, which extends the loss-aware inventory management research to situations involving demand-capacity interactions.

Second, we establish that for both types of shortage costs considered ($ per unit in the unit stockout model and $ per unit per period in the backlogging model), the base-stock levels that minimize the probability of loss are identical and only depend on the cost target and the holding cost rate, but not on the shortage cost. This is important for managers interested in minimizing the probability of loss because we know now that the shortage costs do not play a role in setting optimal base-stock levels.

Third, in our two models that minimize expected loss, we provide exact formulas of the optimal base-stock levels. In addition, we show the impact of cost parameters and demonstrate the influence of capacity utilization levels on the optimal base-stock levels which is often overlooked in related research literature.

Lastly, we consider production-inventory models that minimize expected cost subject to a probability of loss constraint which is not found in the extant literature. Such models are helpful to managers with both expected cost minimization and cost target considerations. We provide solutions featuring upper- and lower-bounds that facilitate a managers’ choice of inventory policies.

The rest of this paper is organized as follows. Section 2 outlines the research setting. Section 3 focuses on minimizing probability of loss, and Section 4 focuses on minimizing expected loss. Section 5 discusses minimizing expected cost with a loss constraint, and Section 6 tests the formulas proposed in Section 3 and Section 4 with simulation experiments. Section 7 provides an extension to a generally-distributed unit production time, and Section 8 briefly summarizes and concludes this paper. Proofs of all lemmas, propositions, and corollaries (as well as three data tables) are included in an appendix.
2. Production-Inventory System Description

We consider a factory consisting of a single-product production system and an inventory location for finished goods. Demand arrivals to the inventory location are a Poisson process with parameter \( \lambda \). The superposition of a large number of stochastic processes usually produces an approximately Poisson process (Cinlar and Agnew, 1968). Since demand arrivals are equal to the superposition of a large number of individual stochastic demands, we assume that a Poisson process is a good model of demand arrivals.

The inventory location places a single-unit order to the production system immediately after receiving a customer order. The production system then produces the item in a unit production time modeled as an exponential random variable with parameter \( \mu \), which is consistent with the case of exponential unit manufacturing time of Arreola-Risa (1996) and the case of exponential processing time with batch size \( Q = 1 \) and zero setup time of Karmarkar (1987). This paper was inspired by our work in a real-world supply chain design project, in which a company produces a large piece of oil and gas extraction equipment that requires manual assembly and welding. At the time of designing the supply chain, the assembly process was expected to be interrupted by quality testing, worker shortages, supply problems, machine breakdowns, and capacity limits (Li and Arreola-Risa, 2017; Shi et al., 2020). Therefore, the project manager decided to use a long-tail distribution for unit production time, and as a result, we chose the exponential distribution as a model of such time. The exponential distribution as a model of unit production time offers the advantage of maximal entropy and minimal information requirement (Jaynes, 1957a,b).

In the production-inventory system, a continuous-review inventory location with base-stock level \( S \) places production orders, whose lead-time is determined by a production system represented by an \( M/M/1 \) queue, due to the demand arrivals being a Poisson process and unit production times following an exponential distribution. Base-stock policies are frequently used in practice. For example, Hewlett-Packard used a base-stock inventory policy for the DeskJet printer supply chain (Lee et al., 1993).

The production-inventory system is a Markov chain that can be characterized by the number of outstanding orders at time \( t \) which will be denoted by \( OO(t) \). Demands will first be satisfied from on-hand inventory if any, and if no on-hand inventory is available, demand will be backlogged. A holding cost is incurred for on-hand inventory. In accordance with prior studies (Arreola-Risa, 1996; Arreola-Risa and DeCroix, 1998; Pan et al., 2009; Hariga, 2010), we consider two types of shortage costs in two different models: A stockout cost measured as dollars per unit, and a backlogging cost.
measured as dollars per unit per period. These shortage costs may arise from penalty clauses in supply contracts or price concessions to maintain customer goodwill. Required raw materials, if any, will be made available in a just-in-time fashion.

For the $M/M/1$ production system, the steady-state probability distribution of the number of outstanding orders $OO$ is given by

$$f_{OO}(k) = (1 - \rho)\rho^k, \ k = 0, 1, \ldots, +\infty$$

where $\rho \equiv \lambda/\mu$ represents the average capacity utilization. This and all other notation is summarized in Table 1. It is easy to show that in the long run, the on-hand inventory level $OH$, the number of stock-outs per period $SO$, and the backorder level $BO$, all depend on the number of outstanding orders $OO$ which follows the geometric distribution shown in Equation (1). It is also not hard to argue that in steady-state, $OH = S - OO + BO$, $OH \cdot SO = 0$, $OH \cdot BO = 0$, and $OH, SO, BO \geq 0$.

Consequently,

$$Pr(OH = k) = \begin{cases} 
Pr(OO = S - k), & 1 \leq k \leq S; \\
Pr(OO \geq S), & k = 0. 
\end{cases}$$

$$Pr(SO = k) = \begin{cases} 
Pr(OH = 0) \cdot Pr(SO = k|OH = 0) = \rho^S \cdot \lambda^k e^{-\lambda}/k!, & k \geq 1; \\
Pr(OO \leq S), & k = 0. 
\end{cases}$$

$$Pr(BO = k) = \begin{cases} 
Pr(OO = S + k), & k \geq 1; \\
Pr(OO \leq S), & k = 0. 
\end{cases}$$

We consider the total cost during a relatively short time horizon which we call a period (e.g., a week) with horizon length $T=1$ and assume steady state at the start of a period. In addition to tractability, this assumption is reasonable because the impact of the initial state on the system state probability distribution diminishes with time (proportional to $1/t$), as shown by Wang and Glynn (2016). Moreover, companies often wait until demand and production become stable to set a structured inventory policy, leveraging on cumulative information about demand patterns and production throughput.

In this paper, the total cost in a period is calculated in a discrete-time manner based on the system state at the beginning of a period. In other words, if there are no units on hand at the beginning of the period, then all demand arrivals in the period are considered to be stockouts, and
if there are units on hand at the beginning of the period, then no demand arrivals during the period will be considered to be stockouts. This calculation method is employed for three reasons:

1. **Excellent Performance.** This calculation method is either exact or a very good approximation to continuous-time cost accounting (a.k.a., full-information) as shown in Section 6, confirming minimal impact of transient dynamics.

2. **Discrete-Time Costing.** The calculation of costs may be discrete in time. For example, a delay of 1.2 weeks may be rounded-up to 2 weeks when calculating penalty in a laddered manner. As another example, if the finance department plans its funding level weekly, holding two units in inventory at the beginning of a week may prompt the finance department to maintain a corresponding loan amount from the bank for the entire week, regardless of inventory changes during the week.

3. **Inadequate Information.** This calculation method may arise from lack of information about the transient behavior of inventory position during the period. In many small and medium enterprises (SMEs), the periodic inventory accounting method is used which means that the firm periodically counts and updates its inventory position in the general ledger (Warren et al., 2011, p.279). Let a period be the horizon under which the finance department plans its funding level and adjust its credit utilization with its creditors. Since information about the specific timing of order arrivals and production completions may not be available, accurate, or timely, the finance department may calculate the costs based on system state at the beginning of a period and thus plans its funding level accordingly.

In the remainder of this paper, we first provide various formulas which are derived from the assumed ‘limited-information’ situation, and then in Section 6 we compare these formulas with ‘full-information’ simulation results.

2.1. **The Unit Stockout Cost Model**

In the Unit Stockout Cost (USC) model, we consider a holding cost $h$ measured in dollars per unit per period and a stockout cost $\pi$ measured in dollars per unit. The unit stockout cost ($\pi$) may represent one-time costs such as expedited delivery or a penalty specified in the supply contract. As an example of a contract penalty, Kroger fines suppliers $500 for each order that is delivered more than 2 days late (McKinsey&Company, 2018). The total cost probability distribution is estimated using the system state at the beginning of a period. In other words, for a period with no on-hand inventory at the beginning, all demand arrivals will be considered as stockouts during the period;
Table 1: Key Notations

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>$\lambda$</td>
<td>Demand arrival rate</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Production rate</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Average capacity utilization $\equiv \lambda/\mu$</td>
</tr>
<tr>
<td>$S$</td>
<td>Base-stock level</td>
</tr>
<tr>
<td>$OO$</td>
<td>(Steady-state) number of outstanding production orders</td>
</tr>
<tr>
<td>$OH$</td>
<td>(Steady-state) on-hand inventory level</td>
</tr>
<tr>
<td>$SO$</td>
<td>(Steady-state) number of stock-outs per period</td>
</tr>
<tr>
<td>$BO$</td>
<td>(Steady-state) backorder level</td>
</tr>
<tr>
<td>$h$</td>
<td>Inventory holding cost per unit per period</td>
</tr>
<tr>
<td>$\pi$</td>
<td>Stockout cost per unit</td>
</tr>
<tr>
<td>$b$</td>
<td>Backlogging cost per unit per period</td>
</tr>
<tr>
<td>$K(\cdot)$</td>
<td>Cost per period</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Cost target</td>
</tr>
<tr>
<td>$\mathbb{E}(\cdot)$</td>
<td>Expected value operator</td>
</tr>
<tr>
<td>$PL_\eta(\cdot)$</td>
<td>Probability of Loss exceeding $\eta$, $PL_\eta(\cdot) = \Pr(K(\cdot) &gt; \eta)$</td>
</tr>
<tr>
<td>$EL_\eta(\cdot)$</td>
<td>Expected Loss exceeding $\eta$, $EL_\eta(\cdot) = \mathbb{E}([K(\cdot) - \eta]^+)$</td>
</tr>
<tr>
<td>$EC(\cdot)$</td>
<td>Expected Cost, $EC(\cdot) = \mathbb{E}(K(\cdot))$</td>
</tr>
<tr>
<td>$f_A(\cdot)$</td>
<td>Probability function of random variable $A$</td>
</tr>
<tr>
<td>$\text{Pois}(A, \lambda)$</td>
<td>Cumulative Poisson Distribution, $\text{Pois}(A, \lambda) = \sum_{n=0}^{A} \frac{\lambda^n e^{-\lambda}}{n!}$</td>
</tr>
<tr>
<td>$\mathbbm{1}(B)$</td>
<td>$= 1$ if $B$ is true and zero otherwise</td>
</tr>
<tr>
<td>$\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$</td>
<td>Rounding up/down to the nearest integer, respectively</td>
</tr>
<tr>
<td>$x_0$</td>
<td>$= \lceil \eta/\pi \rceil \geq 1$</td>
</tr>
<tr>
<td>$k_0$</td>
<td>$= S - \lfloor \eta/h \rfloor$</td>
</tr>
</tbody>
</table>

for a period with on-hand inventory at the beginning, no demand arrivals will be considered as stockouts, and holding cost for the period will be calculated using the initial on-hand inventory. Due to Poisson demand arrivals, we have that $\Pr(SO = k|OH = 0) = \lambda^k e^{-\lambda}/k!$. The total cost in a period is calculated as $K_{USC} = h \cdot T \cdot OH + \pi \cdot SO = h \cdot OH + \pi \cdot SO$ (since $T=1$), which depends
on the random variables $OO = k$ and $SO = x$. Thus

$$
K_{USC} = \begin{cases} 
  h(S - k), & \text{with probability } f_{OO}(k) = (1 - \rho)\rho^k \quad \text{if } 0 \leq k \leq S - 1; \\
  \pi \cdot x, & \text{with probability } f_{SO}(x) = \rho^S \lambda^x e^{-\lambda}/x!, \quad x \in \mathbb{N} \quad \text{if } k \geq S
\end{cases}
$$

(5)

where $\mathbb{N}$ denotes the non-negative integer set. Note that the distribution of $OO$ in Equation (1) is a function of the average capacity utilization, and it is independent of the absolute demand value in a particular period. Also note that the distribution of $K_{USC}$ in Equation (5) does not match any of the well-known probability distributions, which makes it challenging to decipher the base-stock level that minimizes a partial moment of total cost such as probability of loss and expected loss.

### 2.2. The Backlogging Cost Rate Model

In the Backlogging Cost Rate (BCR) model, we consider a holding cost $h$ measured in dollars per unit per period and a backlogging cost $b$ measured in dollars per unit per period. The backlogging cost may represent discounts offered to customers based on how long they wait for their demands to be fulfilled or a penalty specified in the supply contract. Since the costs are calculated based on the system state at the beginning of a period, the total cost probability distribution is estimated using the system state at the beginning of a period. In other words, for a period with backlogged demand at the beginning, backlogging cost for the period will be calculated using the initial backlog position; for a period with on-hand inventory at the beginning, no demand arrivals will be backlogged, and holding cost for the period will be calculated using the initial on-hand inventory position. The total cost in a period is then $K_{BCR} = (h \cdot OH + b \cdot BO)T = h \cdot OH + b \cdot BO$ (since $T=1$). Thus, we have that

$$
K_{BCR} = \begin{cases} 
  h(S - k) & \text{if } 0 \leq k \leq S - 1; \\
  b(k - S) & \text{if } k \geq S
\end{cases}
$$

(6)

in which the probability of the system state being $k$ is given by $(1 - \rho)\rho^k$.

Based on Equations (5) and (6), in Section 3 we respectively minimize the probability of loss for the USC and BCR models. Similarly, based on Equations (5) and (6), in Section 4 we respectively minimize the expected loss for the USC and BCR models.

### 3. Minimizing Probability of Loss

In this section, we minimize the Probability of Loss (PL) with respect to $S$ for both models. The definition of PL is given in Table 1, in which $\eta$ denotes the cost target. The minimization
problem can be stated as:

$$\min PL_\eta(S)$$

s.t. $S \in \mathbb{N}$.

Intuitively, $PL_\eta(S) = 0$ when $\eta \to \infty$ and $PL_\eta(S)$ is close to 1 when $\eta = 0$. Define Probability of Loss entirely due to stockout costs conditioned on $OO \geq S$ as $PLS = Pr(K > \eta|OO \geq S)$. Denote $S^*_{PL}$ as the base-stock level that minimizes PL under this scenario. Our first major result is the theorem below.

**Theorem 1.** For any stockout cost function that results in a $PLS$ that is independent of $S$ and a production system where $OO$ can be modeled as a birth-death process where the birth rate at each system state is lower than the corresponding death rate, we have that $S^*_{PL} = \lfloor \eta/h \rfloor$.

The $M/M/1$ queue that we adopt to model the production system in this paper is a typical birth-death process where the constant birth rate ($\lambda$) is lower than the constant death rate ($\mu$). Therefore, Theorem 1 applies to both the USC and BCR models, which we will formally show in Propositions 1 and 2.

### 3.1. Minimizing Probability of Loss in the Unit Stockout Cost Model

The problem of minimizing the Probability of Loss in the Unit Stockout Cost model will be abbreviated to the PLUSC problem. In the following proposition we establish the base-stock level which minimizes $PL_\eta(S)$.

**Proposition 1.** $S^*_{PLUSC} = \lfloor \eta/h \rfloor$ is the optimal solution to the PLUSC problem, and the minimum value of PL is given by $PL_\eta(S^*_{PLUSC}) = \rho^{\lfloor \eta/h \rfloor} \cdot [1 - P(\lfloor \eta/\pi \rfloor, \lambda)]$.

Based on Proposition 1, we can say that $S^*_{PLUSC}$ is non-decreasing in $\eta$ and is non-increasing in $h$. It is interesting to observe that $S^*_{PLUSC}$ is independent of $\pi$, $\lambda$, and $\rho$. Also note that $PL_\eta(S^*_{PLUSC})$ is decreasing in $\eta$ and increasing in $h$, $\pi$, $\lambda$, and $\rho$.

### 3.2. Minimizing Probability of Loss in the Backlogging Cost Rate Model

The problem of minimizing the Probability of Loss in the BCR Model will be abbreviated to the PLBCR problem. From Equation (6), we have

$$PL_\eta(S) = Pr[h(S - k) > \eta] \cdot 1(hS > \eta) + Pr[b(k - S) > \eta]$$

(7)
which sums up the probability of $k$ when $0 \leq k \leq S - \lfloor \eta/h \rfloor - 1$ or $k \geq S + \lfloor \eta/b \rfloor + 1$. In the following proposition we establish the base-stock level which minimizes $PL_\eta(S)$ in Equation (7).

**Proposition 2.** $S^*_{PLBCR} = \lfloor \eta/h \rfloor$ is the optimal solution to the PLBCR problem, and the minimum value of $PL$ is given by $PL_\eta(S^*_{PLBCR}) = \rho^{\lfloor \eta/h \rfloor + \lfloor \eta/b \rfloor + 1}$.

Note in Proposition 2 that $S^*_{PLBCR}$ is non-decreasing in $\eta$ and non-increasing in $h$. Also note that $PL_\eta(S^*_{PLBCR})$ is decreasing in $\eta$, and increasing in $h$, $b$, $\lambda$, and $\rho$.

3.3. Discussion

It is surprising that in both models the optimal decision is the same according to Theorem 1 and Propositions 1 and 2, namely, $S^*_{PLBCR} = S^*_{PLUSC}$. This indicates that the accuracy of shortage costs should not be a major concern for managers minimizing probability of loss. This is great news for managers since it is always easier to estimate holding costs than to estimate costs arising from unsatisfied demands.

The explanation of the optimal decision being the same for both models is that it is optimal for the decision-maker to minimize the probability of loss by just increasing the units of inventory until the maximum-possible inventory holding cost to be incurred in the period is as close as possible to the cost target. Since cost is tied to profit when revenue is stable (which applies to our setting), this result aligns with prior literature on maximizing the probability of hitting a profit target for a newsvendor (Lau, 1980), but surprisingly the result is also valid in settings with a cost-focus and a capacitated production system. The managerial implication is that although attention should be paid to changes in economic factors, managers can now be sure that their inventory decisions should only depend on the cost target and the holding cost and not on all other economic factors.

4. Minimizing Expected Loss

In this section, we minimize the Expected Loss (EL) with respect to $S$ for both models. The definition of EL is given in Table 1. The minimization problem can be stated as:

$$\min EL_\eta(S)$$

s.t. $S \in \mathbb{N}$.

4.1. Minimizing Expected Loss in the Unit Stockout Cost Model

The problem of minimizing the Expected Loss in the USC model will be abbreviated to the ELUSC problem. Recall that $x_0 = [\eta/\pi]$ and $k_0 = S - [\eta/h]$ as defined in Table 1. The expected
loss per period is

\[ EL_\eta(S) = \rho^S \cdot \sum_{x=0}^{\infty} \left[ (\pi x - \eta) \lambda^x e^{-\lambda x} x! \right] + \sum_{k=0}^{k_0} \left[ h(S - k - \eta/h)(1 - \rho)\rho^k \right] \cdot 1(k_0 \geq 0) \]

\[ = \rho^S \cdot \left\{ -\eta [1 - P(x_0 - 1, \lambda)] + \pi \lambda [1 - P(x_0 - 2, \lambda)] \right\} \]

\[ + \left\{ (1 - \rho^{k_0 + 1})(hS - \eta) - \frac{h}{1 - \rho} \left[ k_0 \rho^{k_0 + 2} - (k_0 + 1)\rho^{k_0 + 1} + \rho \right] \right\} \cdot 1(k_0 \geq 0). \]

The base-stock level candidates to minimize the expected loss per period are provided in the following lemma.

**Lemma 1.** There are three candidates for the value of the optimal base-stock level \( S^*_\text{ELUSC} \). The candidates are respectively called \( S_{\text{ELUSC},1} \), \( S_{\text{ELUSC},2} \), \( S_{\text{ELUSC},3} \) and are given by

1) When \( k_0 < 0 \), \( S_{\text{ELUSC},1} = \lceil \eta/h \rceil - 1 \).

2) When \( k_0 \geq 0 \),

\[
\begin{cases} 
S_{\text{ELUSC},2} = \lceil \eta/h \rceil; \\
S_{\text{ELUSC},3} = \left\lfloor \frac{\ln(h) - \ln\left( u(1 - \rho) + \rho^{1 - \lceil \eta/h \rceil}(1 - \rho)(\eta - h\lceil \eta/h \rceil + h) \right)}{\ln(\rho)} \right\rfloor + 1
\end{cases}
\]  

(8)

in which \( u = -\eta [1 - P(x_0 - 1, \lambda)] + \pi \lambda [1 - P(x_0 - 2, \lambda)] \).

Now we proceed to determine the optimal base-stock level. The following results apply to the three candidates for the optimal base-stock level \( S^*_\text{ELUSC} \):

**Lemma 2.** \( EL_\eta(S_{\text{ELUSC},2}) \leq EL_\eta(S_{\text{ELUSC},1}) \Leftrightarrow S_{\text{ELUSC},3} \geq S_{\text{ELUSC},2} \).

**Lemma 3.** \( S_{\text{ELUSC},3} \geq S_{\text{ELUSC},1} \).

Based on Lemmas 2 and 3, we arrive at the following proposition:

**Proposition 3.** \( S^*_\text{ELUSC} = S_{\text{ELUSC},3} \) is the optimal solution for the ELUSC problem.

It is not hard to show that \( S^*_\text{ELUSC} \) is decreasing in \( \eta \) and increasing in \( \pi \) and \( \rho \). Now let \( S^*_{\text{ECUSC}} \) denote the base-stock level which minimizes expected cost, as opposed to expected loss, in the USC model. It can be easily derived that \( S^*_{\text{ECUSC}} = \left\lfloor \frac{[\ln(h) - \ln(\rho h + (1 - \rho)\pi \lambda)]}{\ln(\rho)} \right\rfloor + 1 \). This expression corresponds to the single-product case of Eq. (15b) in Arreola-Risa (1996). A comparison of \( S^*_\text{ELUSC} \) and \( S^*_{\text{ECUSC}} \) naturally leads to the following corollary:

**Corollary 1.** When \( \eta = 0 \), \( S^*_\text{ELUSC} = S^*_{\text{ECUSC}} \).
Corollary 1 demonstrates the equivalence between $S^*_{ELUSC}$ and $S^*_{ECUSC}$ when the cost target is $0$ and thus all possible cost values are considered in the calculation of ELUSC. Mathematically speaking, when $\eta = 0$, based on the definition of Expected Loss in Table 1, $EL_0(\cdot) = \mathbb{E}\{[K(\cdot) - 0]^+\} = \mathbb{E}\{K(\cdot)\} = EC(\cdot)$.

4.2. Minimizing Expected Loss in the Backlogging Cost Rate Model

The problem of minimizing the Expected Loss in the BCR model will be abbreviated to ELBCR problem. The base-stock level candidates to minimize the expected loss per period are provided in the following lemma.

**Lemma 4.** There are three candidates for the value of the optimal base-stock level $S^*_{ELBCR}$. The candidates are respectively called $S_{ELBCR,1}$, $S_{ELBCR,2}$, $S_{ELBCR,3}$, and are given by

1) When $k_0 < 0$, $S_{ELBCR,1} = \lceil \eta/h \rceil - 1$.

2) When $k_0 \geq 0$,

$$
S_{ELBCR,2} = \lceil \eta/h \rceil;
$$

$$
S_{ELBCR,3} = \left\lfloor \frac{\ln\left(\frac{b}{1-\rho}\right) - \ln\left\{-\left[h[\eta/h] - \eta - \frac{h}{1-\rho}\right] + \rho[\eta/h] + [\eta/h] - 1\right\} + \ln(\rho)}{\ln(\rho)} \right\rfloor + \lceil \eta/h \rceil.
$$

In the following two lemmas, we establish important relationships among the three candidates for the optimal base-stock level.

**Lemma 5.** $EL(S_{ELBCR,2}) \leq EL(S_{ELBCR,1}) \iff S_{ELBCR,3} \geq S_{ELBCR,2}$.

**Lemma 6.** $k_0(S_{ELBCR,3}) \geq -1$.

Building on Lemmas 5 and 6, we establish the optimal base-stock level in Proposition 4.

**Proposition 4.** $S^*_{ELBCR} = S_{ELBCR,3}$ is the optimal solution for the ELBCR problem.

Note that $S^*_{ELBCR}$ is decreasing in $h$ and $\eta$ and increasing in $b$ and $\rho$. Now let $S^*_{ECBCR}$ denote the base-stock level which minimizes expected cost, as opposed to expected loss, in the BCR model. It can be easily derived that $S^*_{ECBCR} = \left\lfloor \frac{\ln(h) - \ln(h+b)}{\ln(\rho)} \right\rfloor$. A comparison of $S^*_{ELBCR}$ and $S^*_{ECBCR}$ naturally leads to the following corollary:

**Corollary 2.** When $\eta = 0$, $S^*_{ELBCR} = S^*_{ECBCR}$.

Corollary 2 demonstrates the equivalence between $S^*_{ELBCR}$ and $S^*_{ECBCR}$ when the cost target is $0$. Note that Corollary 2 mimics Corollary 1.
4.3. Discussion

We have shown that minimizing expected loss is equivalent to minimizing expected cost when \( \eta = 0 \) for the USC and BCR models. We have also established analytical expressions of the optimal base-stock levels for both models. The expressions demonstrate a balance between holding costs and shortage costs, and such balance depends on the cost target value. Figure 1 presents a numerical example of the impact of capacity utilization in the BCR model, where \( h = 1, \eta = 2, \) and \( \mu = 1 \) (hence \( \rho = \lambda/\mu = \lambda \)). In accordance with the range of U.S. capacity utilization statistics (Board of Governors of the U.S. Federal Reserve System, 2018), Figure 1 focuses on capacity utilization levels between 60% and 90%. Note in Figure 1 that generally, a higher capacity utilization in the BCR model amplifies the impact of the backlogging cost on the optimal base-stock level by increasing the probability and magnitude of order backlogs, which necessitates a higher inventory to counter the resulting high costs.

![Figure 1: An example of the impact of capacity utilization on \( S^*_{ELBCR} \)](image)

5. Minimizing Expected Cost with a Probability of Loss Constraint

Having considered the minimization of probability of loss and of expected loss, in this section we turn our attention to the problem of minimizing the expected cost of the production-inventory system subject to a loss constraint. When studying this problem, we follow Jammernegg and Kischka (2013a,b) and incorporate a probability of loss constraint. Constrained-optimization problems such as the one we study are helpful to managers who want to include both cost-minimization and cost-target dimensions.
5.1. Minimizing expected cost with a PL constraint in the USC model

Let $\beta$ denote the upper bound on the probability of loss. The problem of minimizing expected cost subject to a probability of loss constraint in the unit stockout cost model will be abbreviated to the CPUSC Problem and can be stated as:

$$\min \mathbb{E}[K_{USC}(S)]$$

$$s.t. \ PL_\eta(S) \leq \beta$$

$$\beta > 0, \ S \in \mathbb{N}$$

in which $\mathbb{E}[K_{USC}(S)]$ denotes expected cost in the CPUSC problem and $PL_\eta(\cdot)$ is defined in Table 1.

Let $S_{CPUSC}^*$ denote the optimal base-stock level in the CPUSC problem. Recall that $S_{ECUSC}^*$ is the base-stock level which minimizes expected cost in the USC model and $S_{ECUSC}^* = \lfloor \{ \ln(h) - \ln(\rho h + (1 - \rho)\pi \lambda) \} / \ln(\rho) \rfloor + 1$. In Proposition 5 we establish the value of $S_{CPUSC}^*$ and its connection to $S_{ECUSC}^*$.

**Proposition 5.**

$$S_{CPUSC}^* = \begin{cases} 
S_{CPUSC}, & \text{if } S_{ECUSC}^* \leq S_{CPUSC} \\
S_{CPUSC}, & \text{if } S_{CPUSC} \leq S_{ECUSC}^* \\
S_{ECUSC}, & \text{if } S_{CPUSC} \leq S_{ECUSC} \leq S_{CPUSC} \\
\emptyset, & \text{otherwise} 
\end{cases}$$

where

$$S_{CPUSC} = \lfloor \{ \ln(1 - \beta) - \ln(\beta) \} / \ln(\rho) \rfloor$$

$$\overline{S_{CPUSC}} = \lfloor \{ \ln(1 - \beta) - \ln(\beta) + \rho^{-\lfloor \eta/h \rfloor} \} / \ln(\rho) \rfloor$$

Note that $S_{CPUSC}$ and $\overline{S_{CPUSC}}$ respectively can be interpreted as lower and upper bounds on the optimal base-stock level.

5.2. Minimizing expected cost with a PL constraint in the BCR model

The problem of minimizing expected cost subject to a probability of loss constraint in the backlogging cost rate model will be abbreviated to the CPBCR Problem and can be stated as:

$$\min \mathbb{E}[K_{BCR}(S)]$$
\[ \text{s.t. } PL_\eta(S) \leq \beta \]
\[ \beta > 0, \ S \in \mathbb{N} \]
in which \( \mathbb{E}[K_{BCR}(S)] \) denotes expected cost in the CPBCR problem and \( PL_\eta(\cdot) \) is defined in Table 1.

Let \( S_{CPBCR}^* \) denote the optimal base-stock level in the CPBCR problem. Recall that \( S_{ECBCR}^* \) is the base-stock level which minimizes expected cost in the BCR model and 
\[ S_{ECBCR}^* = \left\lfloor \frac{\ln(h) - \ln(h+b)}{\ln(\rho)} \right\rfloor. \]
In Proposition 6 we establish the value of \( S_{CPBCR}^* \) and its connection to \( S_{ECBCR}^* \).

**Proposition 6.**
\[
S_{CPBCR}^* = \begin{cases}
S_{CPBCR}^*, & \text{if } S_{ECBCR}^* \leq S_{CPBCR}^* \\
S_{CPBCR}, & \text{if } S_{CPBCR}^* \leq S_{ECBCR}^* \\
S_{ECBCR}^*, & \text{if } S_{CPBCR}^* \leq S_{ECBCR}^* \leq S_{CPBCR}^* \\
\emptyset, & \text{otherwise}
\end{cases}
\]

where
\[
\left\{ \begin{array}{l}
S_{CPBCR} = \left\lceil \frac{\ln \beta}{\ln \rho} - \left\lfloor \frac{\eta}{b} \right\rfloor - 1 \right\rceil; \\
S_{CPBCR} = \left\lfloor \frac{\ln(1-\beta) - \ln \left( \rho^{-\left\lfloor \eta/h \right\rfloor} - \rho^{\left\lfloor \eta/b \right\rfloor + 1} \right)}{\ln \rho} \right\rfloor.
\end{array} \right.
\]

Note that \( S_{CPBCR}^* \) and \( S_{CPBCR}^* \) respectively can be interpreted as lower and upper bounds on the optimal base-stock level.

5.3. Discussion

Comparing Propositions 5 and 6 it can be observed that the optimal base-stock levels of the CPUSC and CPBCR Problems are both bounded and structurally similar as follows: Unless the expected-cost-minimizing solution satisfies the PL constraint, such constraint determines the optimal base-stock level.

We created Figures 2-4 to explore the impact of the parameters \( \rho, \eta \) and \( \beta \) on the optimal base-stock level. Following Propositions 5 and 6, in addition to the optimal base-stock level, Figures 2-4 include the lower and upper bounds as well as the base-stock level which just minimizes expected cost. Of special importance is the fact that in certain scenarios, the optimal base-stock level does not exist because there is no base-stock level value which satisfies the PL constraint. For example, there is no base-stock level value which satisfies the PL constraint in Figure 2(a) when \( \rho = 0.85 \).
and $\rho = 0.9$. Similarly, there is no base-stock level value which satisfies the PL constraint in Figure 2(c) when $\rho = 0.9$. Figures 3(a), 3(b), 4(a) and 4(c) also contain situations in which there is no base-stock level value which satisfies the PL constraint. The general conclusion from Figures 2-4 is that the impact of the parameters $\rho$, $\eta$ and $\beta$ on the optimal base-stock level is not easily predictable.

We prepared Tables A.1 to A.3 (included in the Appendix) to compare the optimal expected cost in the CPUSC and CPBCR Problems to the expected cost that would be obtained by using the lower and upper bounds of the optimal base-stock levels. As can be seen in Tables A.1 to A.3, and to our surprise, using the bounds in lieu of the optimal base-stock level will yield expected costs that are identical or close to the optimal expected cost.

6. Simulation Experiment

As mentioned earlier, the optimal base-stock levels in Propositions 1 to 4 were derived for a limited information scenario in which the total cost probability distribution is estimated using the system state at the beginning of the period. Specifically, if there are no units on hand at the beginning of the period, then all demand arrivals during the period will be considered to be stock-
outs. At the same time, if there are units on hand at the beginning of the period, then no demand arrivals during the period will be considered stock-outs, and holding cost for the period will be calculated using the units of hand at the beginning of the period.

To test the accuracy of the optimal base-stock levels in Propositions 1 to 4, we conducted an extensive simulation experiment in which we obtained the optimal base-stock levels for a full information scenario in which cost is incurred and registered on a real-time basis. The simulation experiment included the PLUSC, PLBCR, ELUSC and ELBCR Problems. The simulation experiment results are summarized in Tables 2 to 5, where $S^*$ denotes the optimal base-stock level obtained in the simulation. Unless noted otherwise, we use $\mu = 10$ and $h = 1$ for the simulation experiment, and the simulation begins at a random initial state.

Table 2 presents the results for the PLUSC Problem. In all cases the full-information optimal base-stock level is identical to the limited-information optimal base-stock level in Proposition 1. Table 3 presents the results for the PLBCR Problem. In all cases the full-information optimal base-stock level is identical to the limited-information optimal base-stock level in Proposition 2. Table 4 presents the results for the ELUSC Problem. In 34 of the 36 cases the full-information optimal base-stock level is identical to the limited-information optimal base-stock level in Proposition 3. In
the two cases where there is mismatch, the difference is 1 unit. Table 5 presents the results for the ELBCR Problem. In 35 of the 36 cases the full-information optimal base-stock level is identical to the limited-information optimal base-stock level in Proposition 4. In the one case where there is mismatch, the difference is 1 unit. **In summary, the simulation experiment results suggest that the optimal base-stock levels in Propositions 1 to 4 are excellent approximations to the optimal base-stock levels which can be obtained using simulation.**

The great advantage of the optimal base-stock levels in Propositions 1 to 4 is their analytical and closed-form nature. The advantage of getting optimal base-stock levels via simulation is that by default we are using full information to obtain them, but the major disadvantage is that simulation is time consuming and represents a “black box” between the parameters and the optimal base-stock level; i.e., it is hard to understand the impact of the parameters on the optimal solution. As a compromise, we derived two-moment approximations of PL and EL which include transient dynamics within and between periods, as well as full information in the total cost calculations. The derivation of the approximations is included in the Appendix and the resulting PL and EL
approximations are respectively given in equations (A.23) and (A.24). Unfortunately, we can report
that their accuracy is poor when compared to simulation, and that the optimal base-stock levels
in Propositions 1 to 4 are much better that the optimal base-stock levels obtained from equations
(A.23) and (A.24). The results of this comparison are available from the authors upon request.

| Table 2: Simulation experiment results for the PLUSC problem |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|    |      | ρ = 0.7  | ρ = 0.8  | ρ = 0.9  |
|    |      | S*       | S*       | S*       |
|    |      | S_{PLUSC} | S_{PLUSC} | S_{PLUSC} |
| 2  | 5    | 5        | 5        | 5        |
|    | 10   | 10       | 10       | 10       |
| 5  | 5    | 5        | 5        | 5        |
|    | 10   | 10       | 10       | 10       |
| 10 | 5    | 5        | 5        | 5        |
|    | 10   | 10       | 10       | 10       |

| Table 3: Simulation experiment results for the PLBCR problem |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|    |      | ρ = 0.7  | ρ = 0.8  | ρ = 0.9  |
|    |      | S*       | S*       | S*       |
|    |      | S_{PLBCR} | S_{PLBCR} | S_{PLBCR} |
| 2  | 5    | 5        | 5        | 5        |
|    | 10   | 10       | 10       | 10       |
| 5  | 5    | 5        | 5        | 5        |
|    | 10   | 10       | 10       | 10       |
| 10 | 5    | 5        | 5        | 5        |
|    | 10   | 10       | 10       | 10       |

7. Extension to a generally-distributed unit production time

In this section, we explore whether our analytical results continue to hold or remain useful when
the production time is less variable, for example, follows an Erlang-2 distribution. The production
system will therefore be an M/E_2/1 queue.

In addition to the difference between M/E_2/1 and M/M/1 in the simulation experiments, we
have also modified the formula by introducing a parameter \( \sigma \) to replace \( \rho \), since the steady-state
distributions of OO under M/E_2/1 and M/M/1 are different.
Table 4: Simulation experiment results for the ELUSC problem

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<th>( \rho = 0.8 )</th>
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<td>( S^* )</td>
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<td>23</td>
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Table 5: Simulation experiment results for the ELBCR problem

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<th>( \rho = 0.8 )</th>
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<td>6</td>
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<td>10</td>
<td>12</td>
<td>23</td>
</tr>
</tbody>
</table>

We determine the calculation method for the parameter \( \sigma \) based on a heavy-traffic approximation of \( M/G/1 \) queue length initially proposed by Kingman (1961, 1962). Denote \( OO \) under steady-state as \( OO_\infty \). It follows that as \( \rho \uparrow 1 \),

\[
(1 - \rho)OO_\infty \Rightarrow L, \tag{9}
\]

where \( P(L > x) = e^{-2x/(1+c_s^2)} \), \( c_s \) is the squared coefficient of variation of unit production time, and \( c_s = 0.5 \) for the case of Erlang-2 production time. It follows from Equation (9) that

\[
P(OO_\infty \geq k) \approx P(L > (1 - \rho)k) = e^{-2k(1-\rho)/(1+c_s^2)}
\]

and that

\[
P (k \leq OO_\infty < k + 1) \approx P (L > (1 - \rho)k) - P (L > (1 - \rho)(k + 1))
\]

\[
= \left(1 - e^{-2(1-\rho)/(1+c_s^2)}\right) e^{-2k(1-\rho)/(1+c_s^2)}.
\]
Denoting
\[
\sigma = e^{-2(1-\rho)/(1+c^2)}
\]
and noting that \( P(OO_\infty = k) = P(k \leq OO_\infty < k + 1) \), we arrive at
\[
P(OO_\infty = k) \approx (1 - \sigma)\sigma^k,
\]
and hence replacing \( \rho \) by \( \sigma \) in Propositions 1 to 4 should provide a reasonable approximation.

Next, we replaced the exponential distribution with the Erlang-2 distribution (with the same mean) in the production time probability distribution, repeated the simulation experiments in Section 6 with parameters unchanged (except for the production time distribution), and compared the simulation results with our analytical solutions. We used \( \sigma = 0.670 \) for \( \rho = 0.7 \), \( \sigma = 0.766 \) for \( \rho = 0.8 \), and \( \sigma = 0.875 \) for \( \rho = 0.9 \) based on Equation (10).

For the PLUSC and PLBCR problems, there is no change in \( S^* \) values under \( M/E_2/1 \) in comparison to those in Tables 2 and 3 under the same parameters \( (\rho, \pi, b, \eta, h) \). This is not surprising given the absence of \( \rho \) in the expressions of \( S^*_{PLUSC} \) and \( S^*_{PLBCR} \). In other words, \( S^*_{PLUSC} \) and \( S^*_{PLBCR} \) continue to produce optimal or near-optimal base-stock levels for a production system modeled by an \( M/E_2/1 \) queue under full-information, and the change in the probability distribution of unit production time makes no impact.

For the ELUSC and ELBCR problems, there are minor changes in \( S^* \) values under \( M/E_2/1 \) in comparison to those in Tables 4 and 5 under the same parameters \( (\rho, \pi, b, \eta, h) \). We use a superscript \( \circ \) to denote the analytical solutions produced by replacing \( \rho \) by \( \sigma \) in Propositions 1 to 4, and the results are reported in Tables 6 and 7. A perusal of these tables reveal that \( S^\circ_{ELUSC} \) and \( S^\circ_{ELBCR} \) produce optimal or near-optimal base-stock levels for a production system modeled by an \( M/E_2/1 \) queue under full-information, similar to what \( S^*_{ELUSC} \) and \( S^*_{ELBCR} \) do in Section 6.

8. Conclusions and Future Research

We studied a production-inventory system with stochastic demand and capacity under a cost target. Cost includes inventory holding cost and demand shortage cost. We considered three risk-management criteria: Probability of Loss, Expected Loss, and Expected Cost with a Probability of Loss Constraint. For each risk-management criteria we included two types of demand shortage costs: dollars per unit and dollars per unit per period. In turn, using a limited information approximation of the cost probability distribution, we obtained in closed form the optimal base-stock levels for
the resulting six models. Moreover, we tested the accuracy of the optimal base-stock levels in two simulation experiments and concluded that they are extremely accurate.

For the Probability of Loss objective, we found two surprising results. First, the optimal base-stock levels of both types of demand shortage costs are the same. Second, the optimal base-stock levels are a function of the cost target and the holding cost, but they are independent of the demand shortage cost.

For the Expected Loss objective, the optimal base-stock levels of both types of demand shortage costs are bounded and structurally similar. We established that the optimal base-stock levels are decreasing in the cost target and are increasing in both the demand shortage cost and the average capacity utilization.

When minimizing Expected Cost with a Probability of Loss Constraint, we concluded that the impact of the cost target, the upper bound on the probability of loss, and the average capacity utilization on the optimal base-stock levels is not easily predictable.
Future research could include determining capacity decisions for risk-aware managers and finding how inventory and capacity decisions may complement each other in meeting cost targets. Another research avenue could include lab experiments and surveys to pinpoint the preferences and practices of business managers when dealing with cost targets. Extending the model to assembly systems with raw materials and finished goods could be one more promising research direction.
Acknowledgment

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Appendix

Table A.1: Expected cost comparisons for CP models for various values of $\rho$ ($h = 1, \pi = 2, b = 5, \eta = 5, \mu = 10$)

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>EC($S_{CPUSC}$)</th>
<th>EC($S_{CPUSC}^*$)</th>
<th>EC($S_{CPUSC}^*$)</th>
<th>$\beta = 0.4$</th>
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Table A.2: Expected cost comparisons for CP models for various values of $\eta$ ($h = 1$, $\pi = 2$, $b = 5$, $\lambda = 8$, $\mu = 10$)

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<td>7.24</td>
</tr>
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<td>EC($S_{CPUSC}$)</td>
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<tr>
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<td>8.29</td>
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<td>8.03</td>
</tr>
<tr>
<td>7</td>
<td>11.29</td>
<td>8.03</td>
</tr>
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Table A.3: Expected cost comparisons for CP models for various values of $\beta$ ($h = 1$, $\pi = 2$, $b = 5$, $\lambda = 8$, $\mu = 10$)

<table>
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<tr>
<th>$\beta$</th>
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<th>$\eta = 10$</th>
</tr>
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<td>EC($S_{CPUSC}$)</td>
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<tr>
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<td>8.19</td>
</tr>
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<td>7.24</td>
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<td>7.24</td>
</tr>
<tr>
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<td>7.19</td>
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<tr>
<td></td>
<td>EC($S_{CPUSC}$)</td>
<td>EC($S_{CPUSC}$)</td>
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</tr>
<tr>
<td>0.6</td>
<td>16.20</td>
<td>8.03</td>
</tr>
</tbody>
</table>

Proof of Theorem 1

Proof. It follows from $OO$ being a birth-death process where the birth rate at each system state is lower than the corresponding death rate that $f_{OO}(\cdot)$ is a decreasing function. Define $PL_1$ as the probability of loss entirely due to holding costs and $PL_2$ as the probability of loss entirely due to stockout costs. It is evident that as $S$ increases from 0, $PL_1 = 0$ for $S \in [0, \lfloor \eta/h \rfloor]$. In turn, $PL_2 = \sum_{k>S} f_{OO}(k)$ will drop as $S$ increases for $S \in [0, \lfloor \eta/h \rfloor]$.

When $S$ rises beyond $\lfloor \eta/h \rfloor$, $PL_1 \neq 0$. Note that for each unit increase in $S$ beyond $\lfloor \eta/h \rfloor$,
\[ \Delta PL_1 = f_{OO}(S - \lfloor \eta/h \rfloor - 1) \] will be added to \( PL_1 \) such that
\[ PL_1 = \sum_{\lfloor \eta/h \rfloor + 1}^{S} f_{OO}(S - \lfloor \eta/h \rfloor - 1). \]

In turn, for each unit increase in \( S \) beyond \( \lfloor \eta/h \rfloor \), \( PL_2 \) will be reduced by \( \Delta PL_2 = f_{OO}(S) \cdot PLS \).

Since \( \Delta PL_1 > \Delta PL_2 \), it is never optimal to increase \( S \) beyond \( \lfloor \eta/h \rfloor \). Hence, \( S^*_PL = \lfloor \eta/h \rfloor \).

Note that by setting \( b(k - S) > \eta \) and equivalently \( k > S + \eta/b \), we have that
\[ PLS = \sum_{x=\lfloor \eta/\pi \rfloor + 1}^{\infty} \frac{(1 - \rho)\rho^k}{\rho^S} = \rho^{\lfloor \eta/b \rfloor + 1} \] (A.1)
for the PLBCR problem. By setting \( \pi x - \eta > 0 \) and equivalently \( x > \eta/\pi \), we have that
\[ PLS = \sum_{x=\lfloor \eta/\pi \rfloor + 1}^{\infty} [\lambda^x \cdot e^{-\lambda}/x!] = 1 - P(\lfloor \eta/\pi \rfloor, \lambda) \] (A.2)
for the PLUSC problem. PLS in both Equations (A.1) and (A.2) is independent of \( S \).

\[ \textbf{Proof of Proposition 1} \]

\textit{Proof.} From Equation (5) and letting \( \pi x - \eta > 0 \), we have \( x \geq \lfloor \eta/\pi \rfloor + 1 \). Next consider the situation in which holding cost could lead to losses. Letting \( h(S - k) - \eta > 0 \), we have \( k \leq \lfloor S - \eta/h \rfloor - 1 = S - \lfloor \eta/h \rfloor - 1 \). Since \( hS - \eta > 0 \Leftrightarrow S - \lfloor \eta/h \rfloor - 1 \geq 0 \), we have that the Probability of Loss for each period is given by:
\[ PL_\eta(S) = \rho^S \cdot \sum_{x=\lfloor \eta/\pi \rfloor + 1}^{\infty} [\lambda^x \cdot e^{-\lambda}/x!] + \sum_{k=0}^{S - \lfloor \eta/h \rfloor - 1} (1 - \rho)\rho^k \cdot 1(S - \lfloor \eta/h \rfloor - 1 \geq 0) \]
\[ = \rho^S \cdot [1 - P(\lfloor \eta/\pi \rfloor, \lambda)] + (1 - \rho^{S - \lfloor \eta/h \rfloor}) \cdot 1(S - \lfloor \eta/h \rfloor - 1 \geq 0). \]

Intuitively, \( PL_\eta(S) = 0 \) when \( \eta \to \infty \) and \( PL_\eta(S) \) is close to 1 when \( \eta = 0 \). From Theorem 1, we obtain the optimal solution \( S_{PLUSC} = \lfloor \eta/h \rfloor \) and \( PL_\eta(S_{PLUSC}) = \rho^{\lfloor \eta/h \rfloor} \cdot [1 - P(\lfloor \eta/\pi \rfloor, \lambda)] \). \( \square \)

\[ \textbf{Proof of Proposition 2} \]

\textit{Proof.} Note that if \( S - \lfloor \eta/h \rfloor - 1 \geq 0 \), or equivalently, \( hS > \eta \), holding costs could lead to losses.
We have that the Probability of Loss for each period is given by:

\[
PL_\eta(S) = \sum_{S + \lfloor \eta/h \rfloor + 1}^{\infty} (1 - \rho)^k + \sum_{k=0}^{S - \lfloor \eta/h \rfloor - 1} (1 - \rho)^k \cdot 1(S - \lfloor \eta/h \rfloor - 1 \geq 0) \quad (A.3)
\]

\[
= \rho^{S + \lfloor \eta/h \rfloor + 1} + [1 - \rho^{S - \lfloor \eta/h \rfloor}] \cdot 1(S - \lfloor \eta/h \rfloor - 1 \geq 0). \quad (A.4)
\]

From Theorem 1, we obtain the optimal solution \( S_{PLBCR} = \lfloor \eta/h \rfloor \). Hence, we have that \( PL_\eta(S_{PLBCR}) = \rho^{\lfloor \eta/h \rfloor + \lfloor \eta/h \rfloor + 1} \). \( \square \)

**Proof of Lemma 1**

*Proof.* When \( k_0 < 0 \), \( EL_\eta(S) = \rho^S \cdot \{-\eta[1 - P(x_0 - 1, \lambda)] + \pi\lambda[1 - P(x_0 - 2, \lambda)]\} \). Note that \( Pois(-1, \lambda) = 0 \). Letting \( u = -\eta[1 - P(x_0 - 1, \lambda)] + \pi\lambda[1 - P(x_0 - 2, \lambda)] \), we obtain the optimal solution \( S_{ELUSC,1} = \lfloor \eta/h \rfloor - 1 \) and \( EL_\eta(S_{ELUSC,1}) = u\rho^{\lfloor \eta/h \rfloor - 1} \).

When \( k_0 \geq 0 \), defining \( \Delta EL_\eta(S) = EL_\eta(S) - EL_\eta(S - 1) \) we have

\[
EL_\eta(S) = u \cdot \rho^S + \left\{ (1 - \rho^{k_0 + 1}) (hS - \eta) - \frac{h}{1 - \rho} \left[ k_0 \rho^{k_0 + 2} - (k_0 + 1) \rho^{k_0 + 1} + \rho \right] \right\}
\]

\[
= u \cdot \rho^S + (hS - \eta) - (hS - \eta)\rho^{k_0 + 1} - \frac{h}{1 - \rho} \left[ k_0 \rho^{k_0 + 2} - (k_0 + 1) \rho^{k_0 + 1} + \rho \right]
\]

\[
\Delta EL_\eta(S) = -u(1 - \rho)\rho^{S - 1} + h - h\rho^{k_0} + (hS - \eta)(1 - \rho)\rho^{k_0} - h\rho^{k_0} + h
\]

\[
= -u(1 - \rho)\rho^{S - 1} + [(h[\lfloor \eta/h \rfloor] - \eta)(1 - \rho) - h]\rho^{k_0} + h
\]

\[
= \left\{ -u(1 - \rho) + [(h[\lfloor \eta/h \rfloor] - \eta)(1 - \rho) - h]\rho^{1 - \lfloor \eta/h \rfloor} \right\} \rho^{S - 1} + h.
\]

Let \( D_1 = \{ S \in \mathbb{N} : \Delta EL_\eta(S) \leq 0 \} \). If \( D_1 = \emptyset \) we obtain \( S_{ELUSC,case2}^* = S_{ELUSC,2} \); otherwise, we obtain \( S_{ELUSC,case2}^* = \sup D_1 = S_{ELUSC,3} \) since \( \Delta EL_\eta(S) \) is monotonically increasing in \( S \), where \( S_{ELUSC,2} = \lfloor \eta/h \rfloor \) and

\[
S_{ELUSC,3} = \left[ \ln(h) - \ln \left\{ u(1 - \rho) + \rho^{1 - \lfloor \eta/h \rfloor}[(1 - \rho)(\eta - h[\lfloor \eta/h \rfloor]) + h] \right\} \right] + 1.
\]

\( \square \)

**Proof of Lemma 2**
Proof. Since $S_{ELUSC,2} = \lceil \eta / h \rceil$, we have $k_0(S_{ELUSC,2}) = 0$ and

$$EL_\eta(S_{ELUSC,2}) = \rho^{\lfloor \eta / h \rfloor} \cdot u + (1 - \rho)(h \lfloor \eta / h \rfloor - \eta).$$

Next

$$S_{ELUSC,3} \geq S_{ELUSC,2}$$

$$\iff \frac{\ln(h) - \ln \left\{ u(1 - \rho) + \rho^{1-\lfloor \eta / h \rfloor} \cdot \left[ (1 - \rho)(\eta - h \lfloor \eta / h \rfloor) + h \right] \right\}}{\ln(\rho)} \geq \lfloor \eta / h \rfloor - 1$$

$$\iff \ln(h) - \ln \left\{ u(1 - \rho) + \rho^{1-\lfloor \eta / h \rfloor} \cdot \left[ (1 - \rho)(\eta - h \lfloor \eta / h \rfloor) + h \right] \right\} \leq (\lfloor \eta / h \rfloor - 1) \ln(\rho)$$

$$\iff \frac{\rho^{\lfloor \eta / h \rfloor - 1}}{\rho^{\lfloor \eta / h \rfloor - 1} + h} \geq h$$

$$\iff u(1 - \rho) \rho^{\lfloor \eta / h \rfloor - 1} \geq [(1 - \rho)(h \lfloor \eta / h \rfloor - \eta)]$$

$$\iff EL_\eta(S_{ELUSC,2}) \leq EL_\eta(S_{ELUSC,1}).\quad \Box$$

**Proof of Lemma 3**

Proof. Let $A = u(1 - \rho) + \rho^{1-\lfloor \eta / h \rfloor} \cdot (1 - \rho)(\eta + h - h \lfloor \eta / h \rfloor) + h \rho$ where $u$ was defined in the proof of Lemma 1. We have that $A \geq \rho^{1-\lfloor \eta / h \rfloor} \cdot [0 + h \rho] = h \rho^{2-\lfloor \eta / h \rfloor}$. It follows that $S_{ELUSC,3} = \left\lfloor \frac{\ln(h) - \ln(A)}{\ln(\rho)} \right\rfloor + 1 \geq \frac{\ln(h) - \ln(h) - (2 - \lfloor \eta / h \rfloor) \ln(\rho)}{\ln(\rho)} + 1 = \lfloor \eta / h \rfloor - 1 = S_{ELUSC,1}.\quad \Box$

**Proof of Proposition 3**

Proof. It follows from Lemma 2 that when $EL_\eta(S_{ELUSC,2}) \leq EL_\eta(S_{ELUSC,1})$, Case 2 is preferred to Case 1 and therefore $S_{ELUSC,3}$ must be the optimal solution since $S_{ELUSC,3} \geq S_{ELUSC,2}$. Otherwise, when $EL_\eta(S_{ELUSC,2}) > EL_\eta(S_{ELUSC,1})$, we have from Lemma 2 that $S_{ELUSC,3} < S_{ELUSC,2}$ and it follows from Lemma 3 that $S_{ELUSC,3} = S_{ELUSC,1}$ is optimal.\quad \Box

**Proof of Corollary 1**
Proof. Let $\eta = 0$, we have that $x_0 = 0$ and $u = \pi \lambda$. In addition, 

$$S^*_\text{ELUSC} = \left[ \frac{\ln(h) - \ln \{\pi \lambda (1 - \rho) + \rho^{1-\lceil \eta/h \rceil} [(1 - \rho)(\eta - h \lceil \eta/h \rceil) + h]\}}{\ln(\rho)} \right] + 1$$

$$= \left[ \frac{\ln(h) - \ln \{\pi \lambda (1 - \rho) + \rho^{1-\lceil 0 \rceil} [(1 - \rho)(0 - h \lceil 0 \rceil) + h]\}}{\ln(\rho)} \right] + 1$$

$$= \left[ \frac{\ln(h) - \ln[\pi \lambda (1 - \rho) + h \rho]}{\ln(\rho)} \right] + 1$$

$$= S^*_\text{ECUSC}. \quad \Box$$

**Proof of Lemma 4**

Proof. Denote $k_1 = S + \lceil \eta/b \rceil$. If $k_0 < 0$ or equivalently, $hS < \eta$, we have

$$EL_\eta(S) = \sum_{k=k_1}^{\infty} [b(k - S) - \eta](1 - \rho)^k$$

$$= \rho^{k_1}(-\eta - bS) + bk_1 \rho^{k_1} + \frac{b}{1 - \rho} \rho^{k_1+1}$$

$$= \rho^{k_1} \left(-\eta - bS + bk_1 + \frac{b \rho}{1 - \rho}\right)$$

$$= \rho^{S + \lceil \eta/b \rceil} \left(b[\eta/b] - \eta + \frac{b \rho}{1 - \rho}\right).$$

Since $b[\eta/b] - \eta + \frac{b \rho}{1 - \rho} > 0$, we want to increase $S$ as much as possible to minimize $EL_\eta(S)$. In this case, we obtain the optimal solution $S_{ELBCR,1} = \lceil \eta/h \rceil - 1$ and $k_0 = -1$. Hence,

$$EL_\eta(S_{ELBCR,1}) = \left(b[\eta/b] - \eta + \frac{b \rho}{1 - \rho}\right) \cdot \rho^{\lceil \eta/h \rceil + \lceil \eta/b \rceil - 1}.$$

If $k_0 \geq 0$ or equivalently, $hS \geq \eta$, we have

$$EL_\eta(S) = \sum_{k=0}^{k_0} [h(S - k) - \eta](1 - \rho)^k + \sum_{k=k_1}^{\infty} [b(k - S) - \eta](1 - \rho)^k$$

$$= \left(1 - \rho^{k_0+1}\right)(hS - \eta) + hk_0 \rho^{k_0+1} - \frac{h}{1 - \rho} \left(\rho - \rho^{k_0+1}\right)$$

$$+ \rho^{k_1}(-\eta - bS) + bk_1 \rho^{k_1} + \frac{b}{1 - \rho} \rho^{k_1+1}$$

$$= (hS - \eta) - \rho^{k_0+1} \left( hS - \eta - hk_0 - \frac{h}{1 - \rho}\right) - \frac{h \rho}{1 - \rho} + \rho^{k_1} \left( -\eta - bS + bk_1 + \frac{b \rho}{1 - \rho}\right)$$

$$= (hS - \eta) - \rho^{k_0+1} \left(h[\eta/h] - \eta - \frac{h}{1 - \rho}\right) - \frac{h \rho}{1 - \rho} + \rho^{k_1} \left(b[\eta/b] - \eta + \frac{b \rho}{1 - \rho}\right).$$
Note that the above expression applies not only to $k_0 \geq 0$. When $k_0 = -1$, the expression of $EL_\eta(S)$ in Case 2 reduces to that in Case 1. Defining $\Delta EL_\eta(S) = EL_\eta(S) - EL_\eta(S - 1)$, we have

$$\Delta EL_\eta(S) = h + \rho^{\eta/h} \left\{ (h[\eta/h] - \eta)(1 - \rho) - h - \rho^{\eta/h} + \eta/h - 1 \right\}.$$ 

Let $D_2 = \{S \in \mathbb{N} : \Delta EL(S) \leq 0\}$. If $D_2 = \emptyset$ we obtain $S^*_{ELBCR, case2} = S_{ELBCR,2}$; otherwise, we obtain $S^*_{ELBCR, case2} = \sup \{D_2\} = S_{ELBCR,3}$ since $\Delta EL_\eta(S)$ is monotonically increasing in $S$, where $S_{ELBCR,2} = [\eta/h]$ and

$$S_{ELBCR,3} = \left[ -\ln(h) + \ln \left\{ \left[ h - (h[\eta/h] - \eta)(1 - \rho) + \rho^{\eta/h} + \eta/h - 1 \right] \left\{ \left[ b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right] \right\} \right] \right. - \ln(\rho)$$

Note that $EL_\eta(S_{ELBCR,2}) = (1 - \rho)(h[\eta/h] - \eta) + \rho^{\eta/h} + \eta/h \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right)$. Also note $k_1 - k_0 = [\eta/h] + [\eta/b]$.

**Proof of Lemma 5**

**Proof.**

$$S_{ELBCR,3} \geq S_{ELBCR,2}$$

$$\Leftrightarrow \left[ \ln \left( \frac{h}{1-\rho} \right) - \ln \left\{ - \left( h[\eta/h] - \eta - \frac{h}{1-\rho} \right) + \rho^{\eta/h} + \eta/h - 1 \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right) \right\} \right. \right] \geq 0$$

$$\Leftrightarrow \ln \left( \frac{h}{1-\rho} \right) - \ln \left\{ - \left( h[\eta/h] - \eta - \frac{h}{1-\rho} \right) + \rho^{\eta/h} + \eta/h - 1 \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right) \right\} \leq 0$$

$$\Leftrightarrow \ln \left( \frac{h}{1-\rho} \right) \leq \ln \left\{ - \left( h[\eta/h] - \eta - \frac{h}{1-\rho} \right) + \rho^{\eta/h} + \eta/h - 1 \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right) \right\}$$

$$\Leftrightarrow \frac{h}{1-\rho} \leq - \left( h[\eta/h] - \eta - \frac{h}{1-\rho} \right) + \rho^{\eta/h} + \eta/h - 1 \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right)$$

$$\Leftrightarrow 0 \leq - \left( h[\eta/h] - \eta \right) + \rho^{\eta/h} + \eta/h - 1 \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right)$$

$$\Leftrightarrow EL(S_{ELBCR,2}) - EL(S_{ELBCR,1}) \leq 0.$$  

**Proof of Lemma 6**

**Proof.** Let $B = - \left( h[\eta/h] - \eta - \frac{h}{1-\rho} \right) + \rho^{\eta/h} + \eta/h - 1 \left( b[\eta/b] - \eta + \frac{b\rho}{1-\rho} \right)$. Note that $h[\eta/h] - \eta = h([\eta/h] - \eta/h) < h$ and that $b[\eta/b] - \eta = b([\eta/b] - \eta/b) \geq 0$. We have $B \geq \frac{h}{1-\rho} -$
\[ (h[\eta/h] - \eta) > \frac{h}{1-\rho} - h = \frac{h\rho}{1-\rho}. \] Note that \( \rho < 1. \) It follows that \( k_0(S_{ELBCR,3}) = \left\lfloor \frac{\ln(h/\rho) - \ln(B)}{\ln(\rho)} \right\rfloor \geq -1. \]

**Proof of Proposition 4**

Proof. It follows from Lemma 5 that \( S_{ELBCR,3} \) is preferred to both \( S_{ELBCR,1} \) and \( S_{ELBCR,2} \) when \( S_{ELBCR,3} \geq S_{ELBCR,2}. \) And when \( S_{ELBCR,3} < S_{ELBCR,2} \) and thus \( k_0(S_{ELBCR,3}) < 0, \) we have from Lemma 6 that \( k_0(S_{ELBCR,3}) = -1 \) and thus \( S_{ELBCR,3} = S_{ELBCR,1}. \) Therefore, \( S_{ELBCR} = S_{ELBCR,3} \) is the single optimal base-stock level for any given \( \eta. \)

**Proof of Corollary 2**

Proof. When \( \eta = 0, \) we have that \( k_0 = k_1 = S. \) It follows that

\[
S_{ELBCR}^* = \left\lfloor \frac{\ln(h/\rho) - \ln(h[\eta/h] - \eta - \frac{h}{1-\rho}) + \rho^{\lfloor \eta/h \rfloor} + \lfloor \eta/b \rfloor - 1 \left( b[\eta/h] - \eta + \frac{b\rho}{1-\rho} \right)}{\ln(\rho)} \right\rfloor + \left\lfloor \frac{\ln(h/\rho) - \ln(h[0/h] - 0 - \frac{h}{1-\rho}) + \rho^{\lfloor 0/h \rfloor} + \lfloor 0/b \rfloor - 1 \left( b[0/h] - 0 + \frac{b\rho}{1-\rho} \right)}{\ln(\rho)} \right\rfloor + \lfloor 0/h \rfloor
\]

\[
= \left\lfloor \frac{\ln(h) - \ln(h + b)}{\ln(\rho)} \right\rfloor = S_{ECBCR}^*.
\]

**Proof of Proposition 5**

Proof. It is easy to show that \( \mathbb{E}[K_{USC}(S)] \) is convex in \( S. \) We have from Proposition 1 that \( PL_\eta(S) \) decreases in \( S \) until \( S = \lfloor \eta/h \rfloor \) and subsequently increases in \( S. \) Hence, to minimize the expected cost subject to a probability of loss constraint, there are four scenarios:

1. If \( S_{ECUSC}^* \) is very small such that \( PL_\eta(S_{ECUSC}^*) \geq \beta, \) we obtain \( S_{CPUSC}^* = S_{CPUSC} \) by solving \( PL_\eta(S) = \rho^S \cdot [1 - P(\lfloor \eta/\pi \rfloor, \lambda)] \leq \beta \) since \( k_0 < 0 \) (see proof of Proposition 1).

2. If \( S_{ECUSC}^* \) is very large such that \( PL_\eta(S_{ECUSC}^*) \geq \beta, \) we obtain \( S_{CPUSC}^* = S_{CPUSC} \) by solving \( PL_\eta(S) = 1 + \rho^S \cdot [1 - P(\lfloor \eta/\pi \rfloor, \lambda) - \rho^{-\lfloor \eta/h \rfloor}] \leq \beta \) since \( k_0 \geq 0 \) (see proof of Proposition 1).

3. Otherwise, if \( PL_\eta(S_{ECUSC}^*) \leq \beta, \) we arrive at \( S_{CPUSC} \leq S_{ECUSC} \leq S_{CPUSC}. \)
(4) If $S_{CPBCR} > S_{CPBCR}$, the problem has no solution.

**Proof of Proposition 6**

Proof. It is easy to show that $E[K_{BCR}(S)]$ is convex in $S$. We have from Proposition 2 that $PL_{\eta}(S)$ decreases in $S$ until $S = [\eta/h]$ and subsequently increases in $S$. Hence, to minimize the expected cost subject to a probability of loss constraint, there are four scenarios:

1. If $S_{*ECBCR}$ is very small such that $PL_{\eta}(S_{ECBCR}) > \beta$, we obtain $S_{CPBCR} = S_{CPBCR}$ by solving $PL_{\eta}(S) = \rho^{S+[\eta/h]+1} \leq \beta$ since $k_0 < 0$ (see proof of Proposition 2).

2. If $S_{*ECBCR}$ is very large such that $PL_{\eta}(S_{ECBCR}) > \beta$, we obtain $S_{CPBCR} = S_{CPBCR}$ by solving $PL_{\eta}(S) = 1 - \rho^{S-[\eta/h]} + \rho^{S+[\eta/h]+1} \leq \beta$ since $k_0 \geq 0$ (see proof of Proposition 2).

3. Otherwise, if $PL_{\eta}(S_{ECBCR}) \leq \beta$, we arrive at $S_{CPBCR} \leq S_{ECBCR} \leq S_{CPBCR}$.

4. If $S_{CPBCR} > S_{CPBCR}$, the problem has no solution.

**Proof of Two-Moment Approximations: Equations (A.23) and (A.24)**

Proof. To simplify this proof, consider a production-inventory system with both backlogging cost and stockout cost where the system state $OO = i$. Define the (holding and backlogging) cost rate as

$$r_i = \begin{cases} h(S - i), & \text{if } i = 0, 1, 2, \ldots, S; \\ b(i - S), & \text{if } i = S + 1, S + 2, \ldots \end{cases}$$

(A.5)

and the state-transition cost (i.e., stockout cost per unit) as

$$r_{ij} = \begin{cases} \pi, & \text{if } j = i + 1 \text{ and } i, j \in \mathbb{N} \text{ and } j \geq S + 1 \\ 0, & \text{otherwise.} \end{cases}$$

(A.6)

The system state transition matrix is $Q = \{q_{ij}\}$ whose components are defined by

$$q_{ij} = \begin{cases} \lambda, & \text{if } j = i + 1 \text{ and } i, j \in \mathbb{N}; \\ \mu, & \text{if } j = i - 1 \text{ and } i, j \in \mathbb{N}; \\ - \sum_{j \in \mathbb{N}, j \neq i} q_{ij}, & \text{if } j = i \text{ and } i, j \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

(A.7)
Now let $\psi = [\psi_i]$ be the vector of steady-state probabilities, $w = [w_i]$ be the vector of constants
$w_i$, and $\mathbb{N} = \{0, 1, 2, \ldots\}$. Define $R(t) = \mathbb{E}[K(S, t)]$ as the expected total cost up to time $t$ over all
possible initial states, and let $g = g(S)$ denote the average growth rate of $R(t)$ over time when
the make-to-stock system being optimized is in steady-state. We know that $\psi_i = (1 - \rho) \rho^i$ and
$\psi w = \sum_{i=0}^{\infty} \psi_i w_i = 0$ (van Dijk and Sladký, 2006, p. 1047, eq. 3.2). It follows from van Dijk and
Sladký (2006) that the system of equations

$$g = r_i + \sum_{j \in \mathbb{N}, j \neq i} q_{ij} (r_{ij} + w_j - w_i)$$

can be expanded as

$$g = h(S - 0) + \lambda(0 + w_1 - w_0)$$
$$g = h(S - 1) + \lambda(0 + w_2 - w_1) + \mu(0 + w_0 - w_1)$$
$$\vdots$$
$$g = h(S - i) + \lambda(0 + w_{i+1} - w_i) + \mu(0 + w_{i-1} - w_i)$$
$$\vdots$$
$$g = h(S - S) + \lambda(\pi + w_{S+1} - w_S) + \mu(0 + w_{S-1} - w_S)$$
$$g = b(S + 1 - S) + \lambda(\pi + w_{S+2} - w_{S+1}) + \mu(0 + w_S - w_{S+1})$$
$$\vdots$$
$$g = b(i - S) + \lambda(\pi + w_{i+1} - w_i) + \mu(0 + w_{i-1} - w_i)$$
$$\vdots$$

From Equation (A.8) and the normalizing condition $\sum_{i=0}^{\infty} \psi_i = 1$, we have that

$$g = \sum_{i=0}^{\infty} \psi_i g = \sum_{i=0}^{\infty} \psi_i r_i + \lambda \sum_{i=S}^{\infty} \psi_i \pi = h \left( S - \frac{\rho}{1 - \rho} \right) + (h + b) \frac{\rho^{S+1}}{1 - \rho} + \lambda \pi \rho^S.$$  (A.9)

It follows that when $b = 0$,

$$g_{USC}(S) = h \left( S - \frac{\rho}{1 - \rho} \right) + \lambda \pi \rho^S$$

and that when $\pi = 0$,

$$g_{BCR}(S) = h \left( S - \frac{\rho}{1 - \rho} \right) + (h + b) \frac{\rho^{S+1}}{1 - \rho}.$$
Letting \( \mathbf{v} = [v_i] \) be the vector of constants \( v_i \), we have that \( \sum_{i \in N} \psi_i v_i = 0 \) following van Dijk and Sladký (2006, p. 1049, eqn. 3.9). Let

\[
\gamma = u_i + \sum_{j \in N, j \neq i} q_{ij} (v_j - v_i)
\]  

be the variance growth rate, where the vector \( u = [u_i] \) is known as the relative variance vector (van Dijk and Sladký, 2006, p. 1048). Since

\[
\gamma = (1 - \rho) \sum_{i=0}^{\infty} \rho^i u_i,
\]

we will need to find the expression of \( u_i \). It follows from Equation (A.10) that

\[
u_i = \sum_{j \in N, j \neq i} q_{ij} [(r_{ij} + w_j)^2 - w_i^2] + 2(r_i - g)w_i
\]  

(A.11)

where

\[
r_i - g = - \sum_{j \in N, j \neq i} q_{ij} (r_{ij} + w_j - w_i),
\]

(A.12)

and note that

\[
u_0 = \lambda (w_1 + w_0)^2.
\]

(A.13)

After some algebraic manipulation, the expression of \( u_i \) can be simplified to

\[
u_i = \sum_{j \in N, j \neq i} q_{ij} [(r_{ij} + w_j)^2 - w_i^2] - 2w_i \sum_{j \in N, j \neq i} q_{ij} (r_{ij} + w_j - w_i)
\]

\[
= \sum_{j \in N, j \neq i} q_{ij} (r_{ij} + w_j - w_i)^2.
\]

We want to find the expression of \( r_{ij} + w_j - w_i \), and it suffices to find the expression of \( w_j - w_i \).

(i) We will begin with the case \( i \leq S - 1 \). Multiplying each sub-equation (where \( i \) is replaced by \( k \)) in Equation (A.8) by \( \rho^k \) and summing all resulting equations up to the \((i + 1)\)-th sub-equation, we arrive at

\[
g \sum_{k=0}^{i} \rho^k = h \sum_{k=0}^{i} \rho^k (S - k) + \lambda \rho^i (0 + w_{i+1} - w_i)
\]  

(A.14)

which simplifies to

\[
g \cdot \frac{1 - \rho^{i+1}}{1 - \rho} = h \cdot \frac{S \rho^{i+1} (\rho - 1) + (i + 1) \rho^{i+1} - i \rho^{i+2} + (1 - \rho) S - \rho}{(1 - \rho)^2} + \lambda \rho^i (0 + w_{i+1} - w_i)
\]
Thus,

\[ w_{i+1} - w_i = r_{i,i+1} + w_{i+1} - w_i = g \cdot \frac{1 - \rho^{i+1}}{\lambda \rho^i (1 - \rho)} - h \cdot \frac{S \rho^{i+1} (\rho - 1) + (i + 1 - i \rho) \rho^{i+1} + (1 - \rho) S - \rho}{(1 - \rho)^2}. \]  

(A.15)

(ii) We will continue with the case \( i \geq S \). Multiplying each sub-equation (where \( i \) is replaced by \( k \)) in Equation (A.8) by \( \rho^k \) and summing all resulting equations up to the \((i+1)\)-th sub-equation, we arrive at

\[ g \sum_{k=0}^{i} \rho^k = h \sum_{k=0}^{S-1} \rho^k (S - k) + b \sum_{k=S+1}^{\infty} \rho^k (k - S) + \lambda \pi \sum_{k=S}^{\infty} \rho^k + \lambda \rho^i (w_{i+1} - w_i) \]  

which can be simplified to

\[ g \cdot \frac{1 - \rho^{i+1}}{1 - \rho} = h \cdot \frac{\rho^{S+1} - \rho (S + 1) + S}{(1 - \rho)^2} \cdot \rho^{i+1} \left[ \rho (i - S) + S - i - 1 \right] + \rho^{S+1} \right\} + \lambda \pi \cdot \frac{\rho^S - \rho^{i+1}}{1 - \rho} + \lambda \rho^i (w_{i+1} - w_i). \]  

(A.16)

Thus,

\[ w_{i+1} - w_i = r_{i,i+1} + w_{i+1} - w_i = \frac{1}{\lambda \rho^i} \left\{ g \cdot 1 - \rho^{i+1} \right\} - h \cdot \frac{\rho^{S+1} - \rho (S + 1) + S}{(1 - \rho)^2} - b \cdot \frac{\rho^{i+1} (\rho i - \rho S + S - i - 1) + \rho^{S+1}}{(1 - \rho)^2} \right\} \]  

\[ + \pi \cdot \frac{\rho - \rho^{S-i}}{1 - \rho}. \]  

(A.17)

Let \( V(t) = \mathcal{V}[K(S,t)] \) be the conditional variance independent of the initial state, and let \( \gamma = \gamma(S) \) denote the variance growth rate of \( V(t) \) over time when the make-to-stock system being optimized is in steady state. Combining (i) and (ii), we can now proceed to calculate \( \gamma = \gamma(S) \).

\[ \gamma = (1 - \rho) \sum_{i=0}^{\infty} \rho^i u_i \]

\[ = (1 - \rho) \sum_{i=0}^{\infty} \rho^i \lambda \left( r_{i,i+1} + w_{i+1} - w_i \right)^2 + (1 - \rho) \sum_{i=1}^{\infty} \rho^i \mu (w_{i-1} - w_i)^2 \]

\[ = (1 - \rho) \lambda \sum_{i=0}^{S-1} \rho^i (w_{i+1} - w_i)^2 + (1 - \rho) \lambda \sum_{i=S+1}^{\infty} \rho^i (\pi + w_{i+1} - w_i)^2 + (1 - \rho) \sum_{i=1}^{\infty} \rho^i \mu (w_{i-1} - w_i)^2 \]

\[ = (1 - \rho) \lambda \sum_{i=0}^{S-1} \rho^i (w_{i+1} - w_i)^2 + (1 - \rho) \lambda \sum_{i=S}^{\infty} \rho^i (w_{i+1} - w_i)^2 + 2(1 - \rho) \lambda \sum_{i=S}^{\infty} \rho^i \pi (w_{i+1} - w_i) \]
\[ + (1 - \rho)\lambda \sum_{i=S}^{\infty} \rho^i \pi^2 + (1 - \rho) \sum_{i=1}^{S} \rho^i \mu (w_{i-1} - w_i)^2 + (1 - \rho) \sum_{i=S+1}^{\infty} \rho^i \mu (w_{i-1} - w_i)^2 \]
\[ = 2(1 - \rho)\lambda \sum_{i=0}^{S-1} \rho^i (w_{i+1} - w_i)^2 + 2(1 - \rho)\lambda \sum_{i=S}^{\infty} \rho^i (w_{i+1} - w_i)^2 + 2(1 - \rho)\lambda \pi \rho^i (w_{i+1} - w_i) \]
\[ + (1 - \rho)\lambda \pi^2 \sum_{i=S}^{\infty} \rho^i. \]

(A.19)

Substituting Equations (A.15) and (A.18) into Equation (A.19), we obtain the closed-form expression of \( \gamma(S) \):

\[ \gamma(S) = \frac{1}{\lambda} \left\{ \frac{1}{(1 - \rho)^4} \left\{ 2(\pi + 1)\rho^{S+2} \left( b^2 (\rho + \rho^{2S+1} - 2(\rho + 1)\rho^S + 3) + 1 \right) \right. \right. \]
\[ + 2b (\rho (\rho^S - 1) - 1) (h (\rho (\rho^S - 1) - \rho S + S) - \lambda (\rho - 1)\pi (\rho^S - 1)) \]
\[ + (h (\rho - \rho^S + 1 + (\rho - 1)S) + \lambda (\rho - 1)\pi (\rho^S - 1))^2 \} \}
\[ - \frac{1}{(1 - \rho)^4} \left\{ 2\rho \left( h \left( -4\rho - 2(\rho + 3(\rho - 1)S - 2)\rho^{S+1} \right. \right. \right. \]
\[ + 2\rho^{2S+2} + (\rho - 1)^2 S(S + 1) \right. \right. \right. \rho^S (b\rho - \lambda (\rho - 1)\pi) \]
\[ + (\rho^{2S+1} + (\rho - 1)(-2S + 1))\rho^S - 1) \rho^S (b\rho - \lambda (\rho - 1)\pi)^2 \]
\[ + h^2 \rho \left( -\rho - (\rho + 4(\rho - 1)S - 3)\rho^{2S+1} + \rho^{3S+2} \right. \]
\[ + (\rho^S) \left. \right\} + \lambda^2 \pi^2 \rho^S \}. \]

(A.20)

When \( b = 0 \), we obtain

\[ \gamma_{USC}(S) = \frac{1}{\lambda (1 - \rho)^4} \cdot 2(\pi + 1)\rho^{S+2} \left[ h (\rho - \rho^{S+1} + (\rho - 1)S) - \lambda \pi (1 - \rho) (1 - \rho^S) \right]^2 \]
\[ - \frac{2\rho}{\lambda (1 - \rho)^4} \left\{ h \left[ -4\rho - 2(\rho + 3(\rho - 1)S - 2)\rho^{S+1} \right. \right. \]
\[ + 2\rho^{2S+2} + (1 - \rho)^2 S(S + 1) \right. \right. \right. \rho^S \lambda (1 - \rho) \pi \]
\[ + (\rho^{2S+1} + (1 - \rho)(2S + 1)\rho^S - 1) \rho^S [\lambda (1 - \rho)\pi]^2 \]
\[ + h^2 \rho \left[ -\rho - (\rho + 4(\rho - 1)S - 3)\rho^{2S+1} + \rho^{3S+2} \right. \]
\[ + (\rho^S) \left. \right\} + \lambda \pi^2 \rho^S. \]
When $\pi = 0$, we obtain
\[
\gamma_{BCR}(S) = \frac{2\rho^2}{\lambda(1-\rho)}b^2 \left( \rho + (-\rho + 2(\rho - 1)S - 3)\rho^S + 4 \right) + 1 \rho^S \\
- bh \left( (1-\rho)^2 S^2 + (1-\rho)S (\rho + 4\rho^{S+1} + 3) + 2\rho(\rho + 3)(\rho^S - 1) \right) \rho^S \\
+ h^2 \left[ \rho + (\rho^2 + 2\rho - (1-\rho)^2 S^2 + (\rho + 3)(\rho - 1)S - 1) \rho^S \\
+ (-\rho + 2(\rho - 1)S - 3)\rho^{2S+1} + 1 \right].
\] (A.22)

Recall that the time horizon $T = 1$. Since we consider the cost accumulated in a single period, it follows that for both USC and BCR models we could adopt a two-moment normal approximation since Glynn and Haas (2004) show that the accumulated reward in Markov reward chains exhibits asymptotic normality as the time horizon goes to infinity. Consequently,
\[
PL_\eta(S) \approx 1 - \Phi \left( \frac{\eta - g(S)}{\sqrt{\gamma(S)}} \right) \] (A.23)

and
\[
EL_\eta(S) \approx \sqrt{\gamma(S)} \phi \left( \frac{\eta - g(S)}{\sqrt{\gamma(S)}} \right). \] (A.24)

References


Benjaafar, Saif, Mohsen ElHafsi, Tingliang Huang. 2010. Optimal control of a production-inventory system with both backorders and lost sales. Naval Research Logistics 57(3) 252–265.


